Computation of Cavity Flow by Finite Element Method with Finite Spectral Shape Function

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Abstract

The streamfunction-vorticity equations for two-dimentional cavity flow are solved by a new finite element method which uses finite spectral basis functions as shape functions for rectangular elements. Simulations for several cases with different Renolds numbers are performed. Good agreement was obtained in the comparison between the present results with the bench mark solutions.

Keyword: *driven cavity flow, finite spectral method, finite element method*

1. Introduction

During the past decades, Finite Element Methods (FEMs) have been recognized to be powerful tools in the solution of numerous flow problems. More and more methods based on FEM were developed and yielded many satisfactory results in various problems.

The finite spectral method based on non-periodic Fourier integral has succeeded in dealing with spectral methods pointwise [1,2] and has been successfully applied to classical schemes such as NND, ENO etc[3,4]. It is characterized by local property, non-periodicity, orthogonal relation, efficiency and simplicity. This makes it possible to apply finite spectral method to finite element method by using Wang Kernal as Shape functions. In order to verify whether it is applicable, we take driven cavity flow as a test problem and compare the numerical results of this problem with the benchmark solutions.

2. Wang Kernal

Expanding the dicrete pulse function

$$f_j = \begin{cases} 1 & \text{if j=0} \\ 0 & \text{if j=others} \end{cases} \tag{1}$$

by Fourier series and truncated off the Nth term,we obtain

$$W_N(x) = \frac{1}{2N}\sum_{n=-N}^{N} C_n \exp(\frac{i\pi nx}{l}) = \frac{1}{2N}\sum_{n=-N}^{N} C_n \cos(\frac{\pi nx}{l}) \tag{2}$$

where $-l \le x \le l$, and C_n takes 0.5 for $n = \pm N$ but 1 for others. Since it is similar to Dirichlet Kernal which is difined on the infinite interval, we call it Wang Kernal. It is easy to expand one-dimensional Wang Kernal to two dimensions. Two-dimensional finite spectral shape function $W_N(x,y)$ is represented as

$$W_N(x,y) = W_N(x) * W_N(y) \tag{3}$$

Hence $W_N(x,y)$ can be used as basis functions in two-dimensional problems.

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3. Computation of Cavity Flow

3.1 Problem description

The computation of the cavity flow in the square domain has been viewed as one of the standard test problem. The problem consists of a square cavity totally filled with an incompressible viscous fluid and a top wall moving with constant velocity. The problem description is shown in Fig. 1.

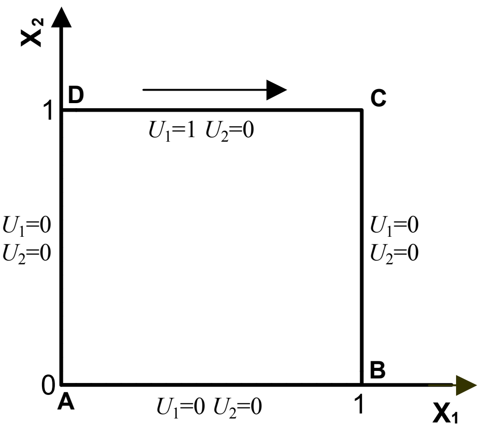


Fig. 1. Driven cavity flow problem description

3.2 Governing equations

For two-dimensional incompressible laminar flows, the Navier-Stokes equations can be written in streamfucntion-vorticity formulation. The streamfunction and vorticity equations can be written in dimensionless form as [5]

$$\frac{\partial^2 \psi}{\partial X_1^2} + \frac{\partial^2 \psi}{\partial X_2^2} = -\omega \tag{4}$$

and

$$\frac{\partial \omega}{\partial t} + U_1 \frac{\partial \omega}{\partial X_1} + U_2 \frac{\partial \omega}{\partial X_2} = \frac{1}{\mathrm{Re}} \left(\frac{\partial^2 \omega}{\partial X_1^2} + \frac{\partial^2 \omega}{\partial X_2^2} \right) \tag{5}$$

where ψ is the streamfunction, $X_i (i = 1, 2)$ are dimensionless Cartesian co-ordinates,

$$U_1 = \frac{\partial \psi}{\partial X_2} \tag{6}$$

and

$$U_2 = -\frac{\partial \psi}{\partial X_1} \tag{7}$$

are the velocity components in the X_1- and X_2- directions respectively,

$$\omega = \frac{\partial U_2}{\partial X_1} - \frac{\partial U_1}{\partial X_2} \tag{8}$$

is the vorticity, t is the time and Re is the Reynolds number.

The boundary conditions of driven cavity flow are given as follow, the walls are no-slip boundaries, the value of the streamfunction is known

$$\psi = 0 \tag{9}$$

since the walls are also streamlines. In the procurement of the wall vorticity, utilize the tangential component of velocity and the streamfunction's value near walls[6].

3.3 Finite element formulation

In the elment, the unknown variables can be approximated by means of the standard expansions

$$\psi \approx \sum_{i=1}^{n} N_i \psi_i \tag{10}$$

$$\omega \approx \sum_{i=1}^{n} N_i \omega_i \tag{11}$$

where N_i are two-dimensional finite spectral basis function, ψ_i and ω_i are nodal values of ψ and ω respectively and n is the number of nodes in the element. Following the Bubnov-Galerkin method, equations (4) and (5) can be written in matrix form as

$$A_{ij}^{(e)} \psi_j - B_{ij}^{(e)} \omega_j = C_i^{(e)} \tag{12}$$

$$D_{ij}^{(e)} \omega_{j,t} + E_{ijk}^{(e)} \psi_j \omega_k + F_{ij}^{(e)} \omega_j = G_i^{(e)} \tag{13}$$

In the above equations

$$A_{ij}^{(e)} = \iint_{\Omega^{(e)}} (N_{i,X_1} N_{j,X_1} + N_{i,X_2} N_{j,X_2}) d\Omega \tag{14}$$

$$B_{ij}^{(e)} = \iint_{\Omega^{(e)}} N_i N_j d\Omega \tag{15}$$

$$C_i^{(e)} = \iint_{\Gamma^{(e)}} N_i q_\psi d\Omega \tag{16}$$

$$D_{ij}^{(e)} = \iint_{\Omega^{(e)}} N_i N_j d\Omega \tag{17}$$

$$E_{ijk}^{(e)} = \iint_{\Omega^{(e)}} N_i (N_{j,X_2} N_{k,X_1} - N_{j,X_1} N_{k,X_2}) \ d\Omega \tag{18}$$

$$F_{ij}^{(e)} = \frac{1}{\mathrm{Re}} \iint_{\Omega^{(e)}} (N_{i,X_1} N_{j,X_1} + N_{i,X_2} N_{j,X_2}) d\Omega \tag{19}$$

$$G_i^{(e)} = \frac{1}{\mathrm{Re}} \iint_{\Gamma^{(e)}} N_i q_\omega d\Omega \tag{20}$$

q_ψ and q_ω are natural boundary conditions of ψ and ω respectively.

3.4 Results and discussion

Using finite element method with finite spectral basis functions, the streamfunction-vorticity equations for two-dimentional cavity flow are uncoupled and solved in sequence on uniformly spaced grids of elements consisting of 21×21 mesh points. The discrete elements are 4-node rectangular elements. Results for $\mathrm{Re} = 100, 400, 1000$ are obtained. The figures of contours of streamfunction and contours of vorticity for $\mathrm{Re} = 400$ shown in Fig. 2 are similar to previous work [5].

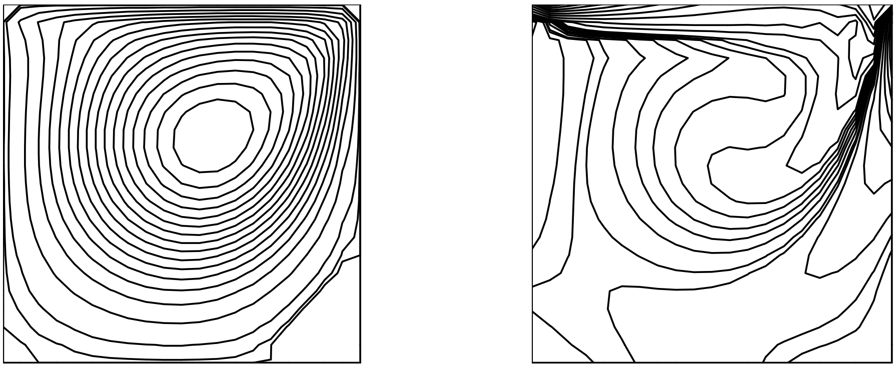
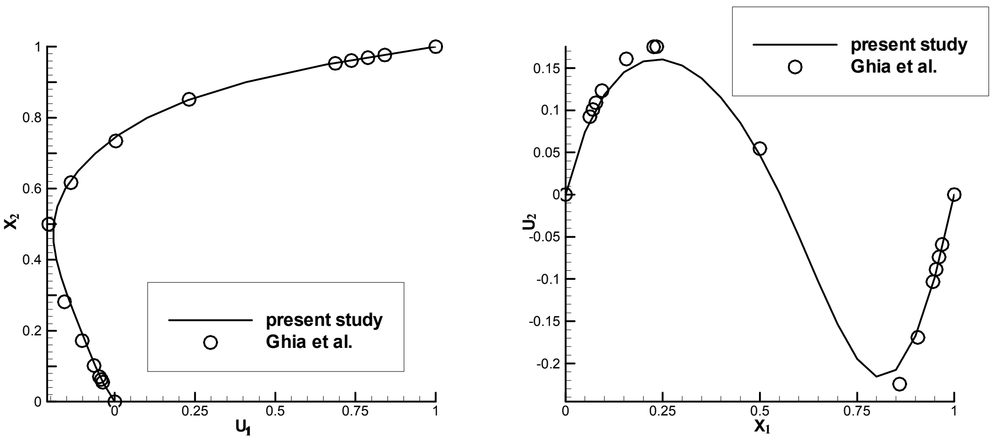


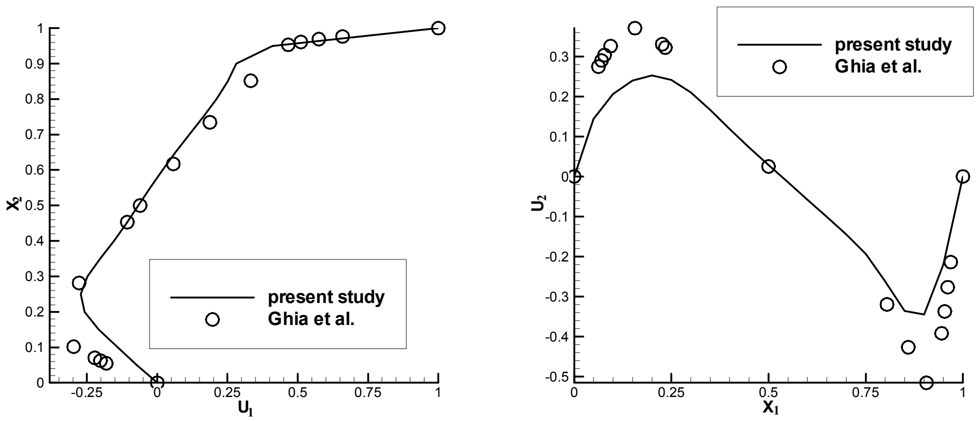
Fig. 2. Contours of streamfunction and contours of vorticity for Re=400

In Fig.3 and Fig.4 we compare the centerline velocity profiles with benchmark solutions [7]. the results are not in well agreement When Re=1000. Consequently we increase mesh points to 31×31 and acquire better results shown in Fig. 5.



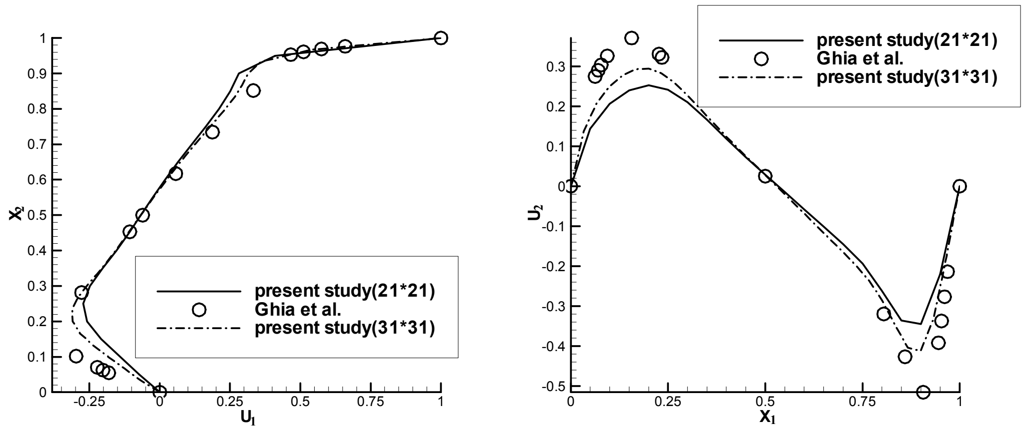
U_1-velocity along vertical centerline U_2-velocity along horizontal centerline

Fig. 3. Velocity profiles along the centreline with 21×21 mesh at Re=100



U_1-velocity along vertical centerline U_2-velocity along horizontal centerline

Fig. 4. Velocity profiles along the centreline with 21×21 mesh at Re=1000



U_1-velocity along vertical centerline U_2-velocity along horizontal centerline

Fig. 5. velocity profiles along the centreline with 31×31 mesh at Re=1000

In Fig. 6. we compare the centerline velocity profiles with previous finite element solutions[8] and find out that They are in well agreement .

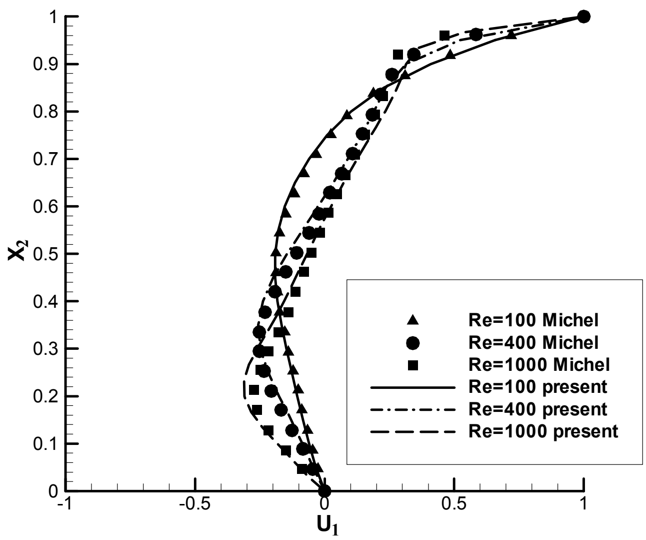


Fig. 6. comparison with previous finite element solution[8]

4. Conclusions

In this paper, we use finite spectral basis functions as interpolation functions for rectangular elements and solve the streamfunction-vorticity equations for two-dimentional cavity flow. Results are in well agreement with previous solutions. These prove that the spectral basis functions are applicable and effective and this method can be used in solving other problems.

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